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ON THE POINTWISE ERGODIC THEOREM ON L^p FOR ARITHMETIC SETS

BY

J. BOURGAIN

IHES, 35 Route de Chartres, 91440 Bures-sur-Yvette, France

ABSTRACT

The purpose of this note is to show how the results of [B] on the pointwise ergodic theorem for L^2-functions may be extended to L^p for certain $p < 2$. More precisely, we give a proof of the almost sure convergence of the means

$$\frac{1}{N} \sum_{1 \leqq n \leqq N} T^{(n^t)} f \qquad (t \geqq 1) \tag{1}$$

given a dynamical system (Ω, B, μ, T) and f of class $L^p(\Omega, \mu)$, $p > (\sqrt{5} + 1)/2$.

1. Reduction

Ergodic means of the form (1) and more general were studied in [B] and almost sure convergence shown assuming f is an $L^2(\mu)$-function. Extending this result to L^p, $p < 2$ requires additional work. We will only consider here sets of the form $\{n^t \mid n = 1, 2, \ldots\}$ $(t > 1)$ for the sake of simplicity, but our argument may be adapted to sets $\{\varphi(n) \mid n = 1, 2, \ldots\}$, φ a polynomial with integer coefficients, as well. Presently, our method, based on interpolation, does not cover the entire range $p > 1$ and the condition $p > (\sqrt{5} + 1)/2$ seems needed.

By standard truncation arguments and the L^2-result proved in [B], it suffices to obtain the maximal inequality on L^p, i.e.,

$$\left\| \sup_N \left(\frac{1}{N} \sum_{1 \leqq n \leqq N} T^{(n^t)} f \right) \right\|_{L^p(\mu)} \leqq C \, \| f \|_{L^p(\mu)}. \tag{2}$$

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This is a problem of a "finite nature" and, as shown in [B], the general case is equivalent to the case of the shift on **Z**. Thus let

$$\mathcal{M} f = \sup_N |f * K_N|$$

where

$$K_N = \frac{1}{N} \sum_{1 \leqq n \leqq N} \delta_{\{n^t\}}. \tag{3}$$

We prove an inequality

$$\| \mathcal{M} f \|_{l^p(\mathbf{Z})} \leqq C(p) \| f \|_{l^p(\mathbf{Z})} \tag{4}$$

provided $p > (1+\sqrt{5})/2$. This restriction seems only technical. The main purpose of this paper is to show that the methods exploited in [B] are not purely L^2.

The argument is based on the same ingredients as for $p = 2$, proved in [B], to which the reader is referred for some of the facts listed below.

2. Exponential sums

The dual group of **Z** is the circle Π and the Fourier transform $\hat{K}_N(\alpha)$, $\alpha \in \Pi$, of K_N is given by the Gauss–Weyl sum

$$\hat{K}_N(\alpha) = \frac{1}{N} \sum_{1 \leqq n \leqq N} e^{-2\pi i n^t \alpha}. \tag{5}$$

Fix $\upsilon = \frac{1}{100}$ and define major arcs $\mathcal{M}_0 = \{\alpha \in \Pi;\ |\alpha| < N^{-t+\upsilon}\}$ and for $1 \leqq a < q < N^{\upsilon}$, $(a, q) = 1$, $\mathcal{M}(q, a) = \{\alpha \in \Pi;\ |\alpha - a/q| < N^{-t+\upsilon}\}$.

LEMMA 1. *If* $\alpha \in \Pi$ *does not belong to a major arc*, $|\hat{K}_N(\alpha)| \leqq N^{-\delta}$ *for some* $\delta = \delta(t) > 0$.

LEMMA 2. *If* $\alpha \in \mathcal{M}(q, a)$, $\alpha = a/q + \beta$, *then* $\hat{K}_N(\alpha) = S(q, a)\hat{k}(N^t\beta) + O(N^{-\upsilon})$ *where*

$$S(q, a) = \frac{1}{q} \sum_{0 \leqq r < q} e^{-2\pi i r^t a/q} \quad \textit{and} \quad k(x) = t^{-1} x^{1/t - 1} \chi_{[0,1]}(x).$$

LEMMA 3. *If* $q = \Pi p_j^{m_j}$ *is the prime decomposition of* q *and* $(a, q) = 1$, *then*

$$|S(q, a)| \leqq C \Pi p_j^{-\overline{m}_j}$$

where we let

$$\bar{m} = 0 \text{ if } m = 0,\ \bar{m} = \tfrac{1}{2} \text{ if } m = 1,\ \bar{m} = 1 \text{ if } 2 \leqq m < t \text{ and } \bar{m} = \frac{m}{t} \text{ if } m \geqq t.$$

3. Partitioning the rationals

Fix $0 < \rho < 1$ and let $\{p_j\}$ be the sequence of consecutive primes. Define for $1 \leqq k \leqq 2^{\rho s}$

$$Q_{s,k} = \prod_{(k-1)2^s < j \leqq k2^s} p_j^{st}. \tag{6}$$

Hence

$$^2\log Q_{s,k} \leqq Cs^2 2^s. \tag{7}$$

Define

$$\mathcal{R}_s = \{\alpha \in \Pi \mid \alpha Q_{s,1} Q_{s,k} \in \mathbf{Z} \text{ for some } 2 \leqq k \leqq 2^{s\rho}\} \tag{8}$$

forming an increasing sequence whose union is $\Pi \cap \mathbf{Q}$. Since

$$\mathcal{R}_s = \mathbf{Z}_{Q_{s,1}} \cup \bigcup_{2 \leqq k \leqq 2^{\rho s}} [\mathbf{Z}_{Q_{s,1} \cdot Q_{s,k}} \setminus \mathbf{Z}_{Q_{s,1}}]$$

where

$$\mathbf{Z}_Q = \{a/Q \mid 0 \leqq a < Q\},$$

$\mathcal{R}_s$ is the disjoint union of $2^{\rho s}$ differences of cyclic subgroups of Π. The next fact is straightforward from Lemma 3 and (8).

LEMMA 4. *If $a/q \in \Pi \setminus \mathcal{R}_s$, then $|S(q, a)| \leqq C2^{-(1+\rho)s/2}$.*

4. Construction of approximate kernels

For $\alpha \in \mathbf{Q} \cap \Pi$, write $\tilde{S}(\alpha) = S(q, a)$ if $\alpha = a/q = 1$. By (7), there is an integer D_s for each s, satisfying

$$^2\log Q_{s,k} < \frac{1}{100}\, {}^2\log D_s \quad (1 \leqq k \leqq 2^{s\rho}) \quad \text{and} \quad \log D_s \leqq Cs^2 2^s. \tag{9}$$

Let φ be a smooth function on $\mathbf{R}$, $\varphi = 1$ on $[-\frac{1}{10}, \frac{1}{10}]$ and $\varphi = 0$ outside $[-\frac{1}{5}, \frac{1}{5}]$. Substitute K_N for a kernel L_N whose Fourier transform is given by

(10) $\hat{L}_N(\alpha) = \hat{k}(N^t\alpha)\varphi(\alpha) + \sum_{s=0}^{\infty} \sum_{\xi \in \mathscr{R}_s \setminus \mathscr{R}_{s-1}} \tilde{S}(\xi)\hat{k}(N^t(\alpha - \xi))\varphi(D_s(\alpha - \xi)).$

There is the following approximation property:

LEMMA 5. *For* $\rho' < \rho$, $\| \hat{K}_N - \hat{L}_N \|_\infty \leqq C(\log N)^{-(1+\rho')/2}$.

PROOF. If $\xi \neq \xi'$ in $\mathscr{R}_s$, then clearly $|\xi - \xi'| > 1/10D_s$. Also, by van der Corput's lemma, $|\hat{k}(\lambda)| \leqq C|\lambda|^{-1/t}$. From Lemma 4, it follows that for $\mathscr{R} \subset \mathscr{R}_s \setminus \mathscr{R}_{s-1}$

$$\left| \sum_{\xi \in \mathscr{R}} \tilde{S}(\xi)\hat{k}(N^t(\alpha - \xi))\varphi(D_s(\alpha - \xi)) \right| \leqq C 2^{-s(1+\rho)/2} \sup_{\xi \in \mathscr{R}} [1 + N^t|\alpha - \xi|]^{-1/t}. \tag{11}$$

Estimating $|\hat{K}_N(\alpha) - \hat{L}_N(\alpha)|$ for $\alpha \in \Pi$ distinguishes the cases α in a major arc and α does not belong to a major arc.

Estimate on major arc. Assume α belongs to the major arc $\mathscr{M}(\xi_0)$. Thus, by Lemma 2,

$$\hat{K}_N(\alpha) = \tilde{S}(\xi_0)\hat{k}(N^t(\alpha - \xi_0)) + O(N^{-v}). \tag{12}$$

Let $\xi_0 \in \mathscr{R}_{s_0} \setminus \mathscr{R}_{s_0 - 1}$. From (11), (12)

$$\begin{aligned} |\hat{K}_N(\alpha) - \hat{L}_N(\alpha)| \leqq\ & C \sum_{s \neq s_0} 2^{-s(1+\rho)/2} \sup_{\substack{\xi \in \mathscr{R}_s \\ \xi \neq \xi_0}} [1 + N^t|\alpha - \xi|]^{-1/t} \\ & + C 2^{-s_0(1+\rho)/2} \sup_{\substack{\xi \in \mathscr{R}_{s_0} \\ \xi \neq \xi_0}} N^{-1}|\alpha - \xi|^{-1/t} \\ & + C 2^{-s_0(1+\rho)/2} |1 - \varphi(D_{s_0}(\alpha - \xi_0))|. \end{aligned} \tag{13}$$

If $\log N < \log D_{s_0} < C2^{(1+\varepsilon)s_0}$ (by (9)), the last two terms in (13) are bounded by $(\log N)^{-(1+\rho)/2+\varepsilon}$. Otherwise, since $|\alpha - \xi_0| < N^{-t+v} < \frac{1}{10}D_{s_0}^{-1}$, the third term vanishes. Writing for $\xi \in \mathscr{R}_s$, $\xi \neq \xi_0$,

$$|\alpha - \xi| \geqq |\xi_0 - \xi| - |\alpha - \xi_0| > 1/N^v D_s - N^{-t+v} \tag{14}$$

it follows in particular for $s = s_0$ that $|\alpha - \xi| > N^{-1-2v}$ and the second term in (13) is bounded by $N^{-1/3}$.

Estimate the first term of (13) as

$$\sum_{2^{(1+\varepsilon)s} < \log N} 2^{-s(1+\rho)/2} \sup_{\substack{\xi \in \mathscr{R}_s \\ \xi \neq \xi_0}} [N^{-1}|\alpha - \xi|^{-1/t}] + (\log N)^{-(1+\rho')/2} \quad (\rho' < \rho). \tag{15}$$

Again by (14), the first term in (15) is at most $CN^{-1/3}$. Hence (13) admits the bound stated in Lemma 3.

Estimate outside major arcs. If α is not in a major arc, then $|\hat{K}_N(\alpha)| < N^{-\delta}$, by Lemma 1 (the H. Weyl estimate). Estimate $|\hat{L}_N(\alpha)|$ by (11),

$$|\hat{L}_N(\alpha)| \leqq C \sum_s 2^{-s(1+\rho)/2} \sup_{\xi\in\mathscr{R}_s} [1 + N^t|\alpha - \xi|]^{-1/t}. \tag{16}$$

By hypothesis, if $\log D_s < \upsilon \log N$, then $|\alpha - \xi| > N^{-t+\upsilon}$ for $\xi \in \mathscr{R}_s$. Otherwise $\log N > Cs^2 2^s$ and $2^{-s(1+\rho)/2} < (\log N)^{-(1+\rho')/2}$. Hence (16) is bounded by $(\log N)^{-(1+\rho')/2}$, which proves Lemma 5.

LEMMA 6. *The $l^1(\mathbf{Z})$-norm of the Fourier transform of the function*

$$\sum_{0\leqq a<q} \tilde{S}(a/q)F(\alpha - a/q), \tag{17}$$

on Π, *is bounded by*

$$2q \sum_{j\in\mathbf{Z}} \sup_{0\leqq x<q} |\hat{F}(jq + x)|. \tag{18}$$

PROOF. The Fourier transform of (17) at the point $x \in \mathbf{Z}$ equals

$$\sum_{0\leqq a<q} \tilde{S}(a/q)e^{2\pi iax/q}\hat{F}(x) = (\#\{0 \leqq r < q \mid x - r^t \in q\mathbf{Z}\})\hat{F}(x).$$

Thus the $l^1(\mathbf{Z})$-norm is bounded by $\Sigma_{0\leqq r<q} \Sigma_{j\in\mathbf{Z}} |\hat{F}(r^t + jq)|$, hence (18).

Taking $F(\alpha) = \hat{k}(N^t\alpha)\varphi(D_s\alpha)$ and $q < D_s$ in Lemma 6, there is a uniform bound

$$\left\| \sum_{0\leqq a<q} \int \tilde{S}(a/q)\hat{k}(N^t(\alpha - a/q))\varphi(D_s(\alpha - a/q))e^{2\pi i\alpha x}d\alpha \right\|_{l^1(\mathbf{Z})} \leqq C. \tag{19}$$

Observe indeed that

$$\hat{F}(x) \sim \int_{-\infty}^{\infty} \left[\int_0^1 y^{1/t-1} e^{-2\pi i(N^t y + x)\alpha} \varphi(D_s\alpha)dy \right] d\alpha$$

$$= D_s^{-1} \int_0^1 y^{1/t-1} \hat{\varphi}(D_s^{-1}(x + N^t y))dy$$

which, since φ is assumed smooth, may be estimated by

$$CD_s^{-1}\int_0^1 y^{1/t-1}\frac{1}{1+\left(\dfrac{x+N^ty}{D_s}\right)^2}dy.$$

Hence, clearly, for $q \leqq D_s$, also

$$\sup_{|x|<q}|\hat{F}(jq+x)| \leqq CD_s^{-1}\int_0^1 y^{1/t-1}\frac{1}{1+\left|\dfrac{jq+N^ty}{D_s}\right|^2}dy$$

and since

$$\sum_{j\in\mathbf{Z}}\left[1+\left(\frac{jq+N^ty}{D_s}\right)^2\right]^{-1} \leqq C\frac{D_s}{q},$$

(18) is bounded by a constant, proving (19).

It is now clear from the construction of the sets $\mathscr{R}_s$ in Section 3 and (19) that

$$\left\|\sum_{\xi\in\mathscr{R}_s}\int_\Pi \tilde{S}(\xi)\hat{k}(N^t(\alpha-\xi))\varphi(D_s(\alpha-\xi))e^{2\pi i\alpha x}d\alpha\right\|_{l^1(\mathbf{Z})} \leqq C2^{s\rho}. \tag{20}$$

Since by Parseval's identity and Lemma 4

$$\begin{aligned}&\left\|\sum_{\xi\in\mathscr{R}_s\setminus\mathscr{R}_{s-1}}\int \tilde{S}(\xi)\hat{k}(N^t(\alpha-\xi))\varphi(D_s(\alpha-\xi))\hat{f}(\alpha)e^{2\pi i\alpha x}d\alpha\right\|_{l^2(\mathbf{Z})}\\ &\leqq C2^{-(1+\rho)s/2}\|f\|_{l^2(\mathbf{Z})}\end{aligned} \tag{21}$$

interpolation between (20), (21) yields, for

$$\frac{1}{\bar{p}}=\frac{1-\bar{\theta}}{2}+\frac{\bar{\theta}}{1}=\frac{1+\bar{\theta}}{2},$$

$$\begin{aligned}&\left\|\sum_{\xi\in\mathscr{R}_s\setminus\mathscr{R}_{s-1}}\int \tilde{S}(\xi)\hat{k}(N^t(\alpha-\xi))\varphi(D_s(\alpha-\xi))\hat{f}(\alpha)e^{2\pi i\alpha x}d\alpha\right\|_{\bar{p}}\\ &\leqq C2^{-s[(1+\rho)(1-\bar{\theta})-2\rho\bar{\theta}]/2}\|f\|_{\bar{p}}.\end{aligned}$$

In order that the sum over $s=0,1,\dots$ in (10) is controlled, it thus suffices to fulfil the condition

$$(1+\rho)(1-\bar{\theta})>2\rho\bar{\theta}. \tag{22}$$

Let $\bar{p}<p<2$,

$$\frac{1}{p}=\frac{1-\theta}{2}+\frac{\theta}{1}=\frac{1+\theta}{2}\equiv\frac{1-\theta'}{2}+\frac{\theta'}{\bar{p}}.$$

Since assuming (22), by definition of L_N

$$\left\| \int \hat{f}(\alpha)\hat{L}_N(\alpha)e^{2\pi i x\alpha}d\alpha \right\|_{\bar{p}} \leqq C \, \| f \|_{\bar{p}},$$

hence

$$\left\| \int \hat{f}(\alpha)(\hat{K}_N-\hat{L}_N)(\alpha)e^{2\pi i x\alpha}d\alpha \right\|_{\bar{p}} \leqq C \, \| f \|_{\bar{p}} \tag{23}$$

and again, by Parseval and Lemma 5,

$$\left\| \int \hat{f}(\alpha)(\hat{K}_N-\hat{L}_N)(\alpha)e^{2\pi i x\alpha}d\alpha \right\|_{2} \leqq C(\log N)^{-(1+\rho')/2} \| f \|_{2}; \tag{24}$$

interpolation between (23), (24) yields

$$\left\| \int \hat{f}(\alpha)(\hat{K}_N-\hat{L}_N)(\alpha)e^{2\pi i x\alpha}d\alpha \right\|_{p} \leqq C(\log N)^{-(1+\rho')(1-\theta')/2} \| f \|_{p}. \tag{25}$$

In evaluating the maximal function $\sup_N |f*K_N|$, f may be taken positive and N of the form 2^k. Hence, by (25),

$$\left\| \sup_k |f*K_{2^k}| \right\|_p \leqq \| \sup |f*L_{2^k}| \, \|_p + (\Sigma \, \| f*(K_{2^k}-L_{2^k}) \|_p^p)^{1/p}, \tag{26}$$

where the second term admits the bound $(\Sigma k^{-(1+\rho')(1-\theta')/2})^{1/p} \| f \|_p$. This leads to the condition

$$\tfrac{1}{2}(1+\rho')(1-\theta')p>1.$$

Since $\theta=\theta'\bar{\theta}$ and ρ' is any number less than ρ, this gives, by (22), the conditions

$$(1+\rho)(1-\theta/\bar{\theta})>1+\theta,$$

hence

$$\rho(1-3\theta)>2\theta.$$

The purpose of the next section is to evaluate the first term in (26).

5. An inequality

In this section, we prove the L^p-version of the inequality given in Section 3 of [B] for the L^2-case.

LEMMA 7. *Let $1 < p < \infty$ and $k \in L^1(\mathbf{R})$ satisfy a maximal inequality*

$$\left\| \sup_{t>0} |f * k_t| \right\|_{L^p(\mathbf{R})} \leqq C(k) \| f \|_{L^p(\mathbf{R})}, \qquad k_t(x) \equiv \frac{1}{t} k\left(\frac{x}{t}\right). \tag{27}$$

Let φ be a smooth function on $\mathbf{R}$ *supported by $[-\frac{1}{2}, \frac{1}{2}]$ and $1 \leqq q < \varepsilon D$ ($\varepsilon = o(1)$). Then*

$$\begin{aligned} &\left\| \sup_{t>0} \left| \sum_{0 \leqq a < q} \int \hat{k}(t\beta) \hat{f}(a/q + \beta) e^{2\pi i x(a/q+\beta)} \varphi(D\beta) d\beta \right| \right\|_{l^p(\mathbf{Z})} \\ &\leqq C'(k) \| f \|_{l^p(\mathbf{Z})}. \end{aligned} \tag{28}$$

We first make some remarks in order to avoid repetition of the same argument. In what follows, ε stands for a small (absolute) constant depending on the choice of φ. Let $0 \leqq u \leqq 1$, then

$$\begin{aligned} &\left\| \int F(\beta)[e^{2\pi i q\beta u} - 1] e^{-2\pi i q\beta y} \varphi(D\beta) d\beta \right\|_{L^p} \\ &\leqq \varepsilon \left\| \int F(\beta) e^{-2\pi i q\beta y} \varphi(D\beta) d\beta \right\|_{L^p}. \end{aligned} \tag{29}$$

Let ψ be a smooth function supported by $[-1, 1]$, $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and let $G(\beta) = F(\beta)\varphi(D\beta)$. Since we may insert a factor $\psi(D\beta)$ in the integrand, the left member of (29) equals

$$q^{-1/p} \| (\hat{G} * \hat{\psi}_D)_{qu} - (\hat{G} * \hat{\psi}_D) \|_p \leqq q^{-1/p} \| \hat{G} \|_p \| \hat{\psi} - \hat{\psi}_{qu/D} \|_1 \leqq \varepsilon q^{-1/p} \| \hat{G} \|_p.$$

LEMMA 8. *With q, D as above*

$$\left\| \int F(\beta) e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{L^p(\mathbf{R})} \sim \left\| \int F(\beta) e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{l^p(\mathbf{Z})}.$$

PROOF OF LEMMA 8. We first prove that $\| \quad \|_{l^p(\mathbf{Z})} \leqq \zeta \| \quad \|_{L^p(\mathbf{R})}$ where ζ is bounded. Let $0 \leqq u < 1$ and write

$$\begin{aligned} &\left\| \int F(\beta) e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{l^p} \\ \leqq &\left\| \int F(\beta) e^{2\pi i \beta q (y+u)} \varphi(D\beta) d\beta \right\|_{l^p} + \left\| \int F(\beta)[1 - e^{2\pi i \beta q u}] e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{l^p}. \end{aligned}$$

Integrating the pth power of the first term in u, $\| \quad \|_{L^p(\mathbf{R})}$ is obtained. For fixed u, estimate the second term

$$\zeta \left\| \int F(\beta)[1 - e^{2\pi i \beta q u}] e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{L^p} \leqq \varepsilon \zeta \left\| \int F(\beta) e^{2\pi i q \beta y} \varphi(D\beta) \right\|_{L^p}$$

using (29). Hence $\zeta \leqq 1 + \varepsilon\zeta$ and we may take $\zeta = 2$.

To prove the converse inequality, write

$$\| \quad \|_{L^p(\mathbf{R})} \leqq \| \quad \|_{l^p(\mathbf{Z})} + \left\{ \int_0^1 \left\| \int F(\beta) e^{2\pi i \beta q y} [1 - e^{2\pi i \beta q u}] \varphi(D\beta) d\beta \right\|_{l^p}^p du \right\}^{1/p}.$$

Appealing to the converse inequality and (29), the inner l^p-norm is clearly bounded by $2\varepsilon \| \int F(\beta) e^{2\pi i \beta q y} \varphi(D\beta) d\beta \|_{L^p}$, proving the inequality.

Proof of Lemma 7. Writing $x \in \mathbf{Z}$ as $x = yq + z$, $z = 0, 1, \ldots, q-1$, the left member of (28) equals

$$\left\{ \sum_{0 \leqq z < q} \left\| \sup_{t > 0} \left| \int \hat{k}(t\beta) F_z(\beta) e^{2\pi i q \beta y} \varphi(D\beta) d\beta \right| \right\|_{l^p(\mathbf{Z}, y)}^p \right\}^{1/p} \tag{30}$$

denoting

$$F_z(\beta) = \sum_{0 \leqq a < q} \hat{f}(a/q + \beta) e^{2\pi i z (a/q + \beta)}.$$

As in the proof of Lemma 8, denote ζ the *a priori* best constant in the inequality

$$\begin{aligned} &\left\| \sup_{t > 0} \left| \int \hat{k}(t\beta) F(\beta) e^{2\pi i q \beta y} \varphi(D\beta) d\beta \right| \right\|_{l^p} \\ &\leqq \zeta \left\| \int F(\beta) e^{2\pi i \beta q y} \varphi(D\beta) d\beta \right\|_{l^p}. \end{aligned} \tag{31}$$

For $0 \leqq u < 1$, write again

$$\begin{aligned} &\sup_{t > 0} \left| \int \hat{k}(t\beta) F(\beta) e^{2\pi i q \beta y} \varphi(D\beta) d\beta \right| \\ &\leqq \sup_{t > 0} \left| \int \hat{k}(t\beta) F(\beta) e^{2\pi i q \beta (y+u)} \varphi(D\beta) d\beta \right| \\ &\quad + \sup_{t > 0} \left| \int \hat{k}(t\beta) F(\beta) e^{2\pi i q \beta y} [e^{2\pi i q \beta u} - 1] \varphi(D\beta) d\beta \right|. \end{aligned} \tag{32}$$

Integrating the pth power of the first term in u on $[0, 1]$ gives, by Lemma 8, an estimate

$$q^{-1/p}\left\|\sup_{t>0}\left|k_t*\left[\int F(\beta)e^{2\pi i\beta x}\varphi(D\beta)d\beta\right]\right|\right\|_{L^p}$$
$$\leqq C(k)q^{-1/p}\left\|\int F(\beta)e^{2\pi i\beta x}\varphi(D\beta)d\beta\right\|_{L^p}$$
$$=C(k)\left\|\int F(\beta)e^{2\pi i\beta q y}\varphi(D\beta)d\beta\right\|_{L^p}$$
$$\sim C(k)\left\|\int F(\beta)e^{2\pi i\beta q y}\varphi(D\beta)d\beta\right\|_{l^p}.$$

The l^p-norm of the second term in (32), for fixed $u\in[0, 1]$, is bounded by

$$\zeta\left\|\int F(\beta)[e^{2\pi i q\beta u}-1]e^{2\pi i q\beta y}\varphi(D\beta)d\beta\right\|_{l^p}$$
$$\leqq 2\zeta\left\|\int F(\beta)[e^{2\pi i q\beta u}-1]e^{2\pi i q\beta y}\varphi(D\beta)d\beta\right\|_{L^p}$$
$$\leqq \varepsilon\zeta\left\|\int F(\beta)e^{2\pi i q\beta y}\varphi(D\beta)d\beta\right\|_{L^p}$$
$$\leqq 2\varepsilon\zeta\left\|\int F(\beta)e^{2\pi i q\beta y}\varphi(D\beta)d\beta\right\|_{l^p}$$

using twice Lemma 8 and (28).

Hence, we proved that $\zeta\leqq C'(k)+2\varepsilon\zeta$ and thus $\zeta\leqq C''(k)$.

Estimating (30) and applying (31) with $F=F_z$ $(z=0,\dots,q-1)$ now easily yields Lemma 7.

6. End of the proof

To complet the proof of (2), it remains to estimate the first term of (26), $1/p=(1+\theta)/2$ where ρ, θ satisfy

$$\rho(1-3\theta)>2\theta.$$

For fixed s, estimate $\Lambda_{p,s}$, the best constant in the inequality

$$\left\|\sup_{\lambda>0}\left|\sum_{\xi\in\mathscr{R}_s\setminus\mathscr{R}_{s-1}}\int\hat f(\xi+\beta)\hat k(\lambda\beta)\varphi(D_s\beta)e^{2\pi i x(\xi+\beta)}d\beta\right|\right\|_{l^p}$$
$$\leqq\Lambda_{p,s}\,\|f\|_{l^p}. \tag{33}$$

Since, from the triangle inequality, we have

$$\left\| \sup_N |f * L_N| \right\|_{l^p} \leqq \left(\sum_s \Lambda_{p,s} \right) \| f \|_{l^p}, \tag{34}$$

the additional request will be

$$\Sigma \Lambda_{p,s} < \infty. \tag{35}$$

By construction, $\mathscr{R}_s \setminus \mathscr{R}_{s-1}$ is the disjoint union of at most 2^{sp} subsets Γ of Π, where each Γ is a difference of cyclic subgroups. Let ψ be a smooth function on $\mathbf{R}$, $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\omega = 0$ outside $[-1, 1]$. Defining

$$F(x) = \sum_{\xi \in \Gamma} \hat{S}(\xi) \int \hat{f}(\alpha) e^{2\pi i \alpha x} \psi(D_s(\alpha - \xi)) d\alpha$$

one clearly has

$$\begin{aligned} &\sum_{\xi \in \Gamma} \hat{S}(\xi) \int \hat{f}(\xi + \beta) \hat{k}(\lambda\beta) e^{2\pi i x(\xi + \beta)} \varphi(D_s \beta) d\beta \\ &\quad = \sum_{\xi \in \Gamma} \int \hat{F}(\xi + \beta) \hat{k}(\lambda\beta) e^{2\pi i x(\xi + \beta)} \varphi(D_s \beta) d\beta. \end{aligned} \tag{36}$$

Note that for $k(x) \sim x^{-1+1/t} \chi_{[0,1]}(x)$ on $\mathbf{R}$, (27) holds for all $1 < p \leqq \infty$, as an easy consequence of the standard Hardy–Littlewood maximal inequality.

Hence, by Lemma 7, for $1 < r \leqq 2$

$$\begin{aligned} &\left\| \sup_{\lambda > 0} \left| \sum_{\Gamma} \int \hat{F}(\xi + \beta) \hat{k}(\lambda\beta) e^{2\pi i x(\xi + \beta)} \varphi(D_s \beta) d\beta \right| \right\|_r \\ &\quad \leqq C(r) \left\| \sum_{\Gamma} \hat{S}(\xi) \left[\int \hat{f}(\alpha) e^{2\pi i \alpha x} \psi(D_s(\alpha - \xi)) d\alpha \right] \right\|_r. \end{aligned} \tag{37}$$

For $r = 2$, Lemma 4 and Parseval give an estimate

$$2^{-(1+\rho)s/2} \left\{ \sum_{\xi \in \Gamma} \int |\hat{f}(\alpha)|^2 |\psi(D_S(\alpha - \xi))|^2 d\alpha \right\}^{1/2}.$$

Hence, since the functions $\psi(D_s(\cdot - \xi))$, $\xi \in \mathscr{R}_s \setminus \mathscr{R}_{s-1}$, are disjointly supported, it follows from Cauchy–Schwartz, by the triangle inequality,

$$\begin{aligned} &\left\| \sup_{\lambda > 0} \left| \sum_{\xi \in \mathscr{R}_s \setminus \mathscr{R}_{s-1}} \hat{S}(\xi) \int \hat{f}(\xi + \beta) \hat{k}(\lambda\beta) e^{2\pi i x(\xi + \beta)} \varphi(D_s \beta) d\beta \right| \right\|_2 \\ &\quad \leqq C 2^{sp/2 - (1+\rho)s/2} \| f \|_2. \end{aligned} \tag{38}$$

Since, by Lemma 6 applied to $F(\beta)=\psi(D_s\beta)$, the Fourier transform of $\Sigma_{\xi\in\Gamma}\tilde{S}(\xi)\psi(D_s(\alpha-\xi))$ is an $l^1(\mathbf{Z})$-bounded function, (37) may be estimated by $C\,\|f\|_r$. Thus

$$\left\|\sup_{\lambda>0}\left|\sum_{\xi\in\mathcal{R}_s\setminus\mathcal{R}_{s-1}}\tilde{S}(\xi)\int\hat{f}(\xi+\beta)\hat{k}(\lambda\beta)e^{2\pi ix(\xi+\beta)}\varphi(D_s\beta)d\beta\right|\right\|_r \tag{39}$$
$$\leqq 2^{ps}C(r)\,\|f\|_r$$

by (36), (37) and the triangle inequality.

Writing $1/p=(1-\upsilon)/2+\upsilon/r$, interpolation between (38), (39) gives

$$\Lambda_{p,s}\leqq C(r)2^{-s(1-\upsilon)/2}2^{sp\upsilon}$$

leading to the condition

$$1-\upsilon>2p\upsilon \tag{40}$$

where υ is any number chosen larger than θ. Hence

$$1-\theta>2p\theta \tag{41}$$

ensures (35).

The existence of p fulfilling both (27) and (41) leads to the restriction

$$\theta<\tfrac{1}{3},\quad \theta^2+4\theta-1<0,\quad \text{hence}\quad \theta<\sqrt{5}-2.$$

This restriction is equivalent to

$$\frac{1}{p}<\frac{\sqrt{5}-1}{2}\quad\text{or}\quad p>\frac{\sqrt{5}+1}{2}.$$

This completes the proof of the result stated in the abstract.

References

[B] J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Isr. J. Math. **61** (1988), 39–72, this issue.